Modular structures on trace class operators and applications to Landau levels

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S Twareque Ali[1]**, F Bagarello**[2,4] **and G Honnouvo**[3]

[1] Department of Mathematics and Statistics, Concordia University, Montréal, Québec, H3G 1M8, Canada
[2] Dipartimento di Metodi e Modelli Matematici, Facoltà di Ingegneria, Università di Palermo, I-90128 Palermo, Italy
[3] Department of Mathematics and Statistics, McGill University, Burnside 1021, 805 Sherbrooke W. Montreal, Québec, H3A 2K6, Canada

E-mail: stali@mathstat.concordia.ca and bagarell@unipa.it

Abstract

The energy levels, generally known as the Landau levels, which characterize the motion of an electron in a constant magnetic field, are those of the one-dimensional harmonic oscillator, with each level being infinitely degenerate. We show in this paper how the associated von Neumann algebra of observables displays a modular structure in the sense of the Tomita–Takesaki theory, with the algebra and its commutant referring to the two orientations of the magnetic field. A Kubo–Martin–Schwinger state can be built which, in fact, is the Gibbs state for an ensemble of harmonic oscillators. Mathematically, the modular structure is shown to arise as the natural modular structure associated with the Hilbert space of all Hilbert–Schmidt operators.

PACS numbers: 71.70.Di, 03.50.De, 41.20.−q, 03.65.Fd

1. Introduction

The motion of an electron in a constant electromagnetic field is a well-known problem in atomic physics. Quantum mechanically, the energy levels of such a system, which are generally known as the Landau levels (see, for example, [8]), are linearly spaced, with each level being infinitely degenerate. Indeed, the energy levels are exactly those of the harmonic oscillator, with infinite degeneracy at each level. The Hamiltonian of the system can be written as the sum of two oscillator Hamiltonians, together with an interaction part, which is an angular momentum term. It turns out that the diagonalized Hamiltonian resembles that of a single harmonic oscillator, with infinite multiplicity at each level. If the sense of the magnetic field

[4] Home page: www.unipa.it/~bagarell

is reversed, one obtains a second Hamiltonian, similar to the first but commuting with it. Both these Hamiltonians can be written in terms of two pairs of mutually commuting oscillator-type creation and annihilation operators, which then generate two von Neumann algebras which mutually commute, and in fact are commutants of each other. This leads to the existence of a modular structure, in the sense of the Tomita–Takesaki theory [16]. The invariant state of the theory turns out to be the Gibbs state for an ensemble of harmonic oscillators; the modular operator, giving the time-evolution under which this state is invariant, is directly obtained from the interaction Hamiltonian and the modular conjugation operator simply interchanges the two possible orientations of the magnetic field. Preliminary discussions of some aspects of the theory presented in this paper have been given in [1–3] and [9]. However, here we present a unified discussion along with a physical interpretation and explore holomorphic aspects of the theory, its connection to families of orthogonal polynomials (Hermite and complex Hermite) and to various related families of coherent states.

The rest of the paper is organized as follows. In section 2 we briefly recall the main features of the Tomita–Takesaki modular theory of von Neumann algebras; in section 3 we work out a simple example of this theory in the space of Hilbert–Schmidt operators on a Hilbert space; in sections 4–6 we give a detailed analysis of the problem of the electron in a constant magnetic field in the light of the modular theory, bringing out the physical meaning of its various mathematical ingredients. In section 7 we look at some associated families of coherent states. Finally in section 8 we make some closing comments. Certain mathematical properties of von Neumann algebras, which are required in the paper, are collected in the appendix.

2. Summary of the mathematical theory

This section is devoted to a quick review of the Tomita–Takesaki modular theory of von Neumann algebras, to the extent that is needed in this paper. Details and proofs of statements may be found, for example, in [14–16]. Some basic definitions and notions about von Neumann algebras are listed in the appendix. Let $\mathfrak{A}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ and $\mathfrak{A}'$ its commutant. Let $\Phi \in \mathfrak{H}$ be a unit vector which is cyclic and separating for $\mathfrak{A}$. Then the corresponding state φ on the algebra, $\langle \varphi; A \rangle = \langle \Phi \mid A\Phi \rangle$, $A \in \mathfrak{A}$, is faithful and normal. Consider the antilinear map

$$S : \mathfrak{H} \longmapsto \mathfrak{H}, \qquad SA\Phi = A^*\Phi, \qquad \forall A \in \mathfrak{A}. \tag{2.1}$$

Since Φ is cyclic, this map is densely defined and in fact it can be shown that it is closable. We denote its closure again by S and write its polar decomposition as

$$S = J\Delta^{\frac{1}{2}} = \Delta^{-\frac{1}{2}}J, \qquad \text{with} \quad \Delta = S^*S. \tag{2.2}$$

The operator Δ, called the *modular operator*, is positive and self-adjoint. The operator J, called the *modular conjugation operator*, is antiunitary and satisfies $J = J^*$, $J^2 = I_{\mathfrak{H}}$. Note that the antiunitarity of J implies that $\langle J\phi \mid J\psi \rangle = \langle \psi \mid \phi \rangle$, $\forall \phi, \psi \in \mathfrak{H}$.

Since Δ is self-adjoint, using its spectral representation, we see that for $t \in \mathbb{R}$, the family of operators $\Delta^{-\frac{i}{t}\beta}$, for some fixed $\beta > 0$, defines a unitary family of automorphisms of the algebra $\mathfrak{A}$. Denoting these automorphisms by $\alpha_\varphi(t)$, we may write

$$\alpha_\varphi(t)[A] = \Delta^{\frac{i}{t}\beta} A \Delta^{-\frac{i}{t}\beta}, \qquad \forall A \in \mathfrak{A}. \tag{2.3}$$

Thus, they constitute a strongly continuous one-parameter group of automorphisms called the *modular automorphism group*. Denoting the generator of this one-parameter group by $\mathbf{H}_\varphi$, we get

$$\Delta^{-\frac{i}{t}\beta} = e^{it\mathbf{H}_\varphi} \qquad \text{and} \qquad \Delta = e^{-\beta\mathbf{H}_\varphi}. \tag{2.4}$$

It can then be shown that the state φ is invariant under this automorphism group:

$$e^{-\beta \mathbf{H}_\varphi} \Phi = \Phi, \qquad \Delta^{\frac{i}{i}\beta} \mathfrak{A} \, \Delta^{-\frac{i}{i}\beta} = \mathfrak{A}, \tag{2.5}$$

and the antilinear map J interchanges $\mathfrak{A}$ with its commutant $\mathfrak{A}'$:

$$J\mathfrak{A}J = \mathfrak{A}'. \tag{2.6}$$

Finally, the state φ can be shown to satisfy the KMS (*Kubo–Martin–Schwinger) condition*, with respect to the automorphism group $\alpha_\varphi(t)$, $t \in \mathbf{R}$, in the following sense. For any two $A, B \in \mathfrak{A}$, the function

$$F_{A,B}(t) = \langle \varphi; A\alpha_\varphi(t)[B] \rangle \tag{2.7}$$

has an extension to the strip $\{z = t + iy \mid t \in \mathbb{R}, y \in [0, \beta]\} \subset \mathbb{C}$ such that $F_{A,B}(z)$ is analytic in the open strip $(0, \beta)$ and continuous on its boundaries. Moreover, it also satisfies the boundary condition (*at an inverse temperature β*)

$$\langle \varphi; A\alpha_\varphi(t + i\beta)[B] \rangle = \langle \varphi; \alpha_\varphi(t)[B]A \rangle, \qquad t \in \mathbb{R}. \tag{2.8}$$

3. A simple example of the theory

A simple example of the Tomita–Takesaki theory and its related KMS states can be built on the space of Hilbert–Schmidt operators on a Hilbert space. The set of Hilbert–Schmidt operators is itself a Hilbert space, and there are two preferred algebras of operators on it, which carry the modular structure. The presentation here follows that in [1] (chapter 8, section 4). A detailed application of this structure to Landau levels is discussed in section 4, which extends some recent work reported in [2].

Again, let $\mathfrak{H}$ be a (complex, separable) Hilbert space of dimension N (finite or infinite) and $\{\zeta_i\}_{i=1}^N$ an orthonormal basis of it ($\langle \zeta_i \mid \zeta_j \rangle = \delta_{ij}$). We denote by $\mathcal{B}_2(\mathfrak{H}) \simeq \mathfrak{H} \otimes \overline{\mathfrak{H}}$ the space of all Hilbert–Schmidt operators on $\mathfrak{H}$. This is a Hilbert space with the scalar product

$$\langle X \mid Y \rangle_2 = \mathrm{Tr}[X^*Y].$$

The vectors

$$\{X_{ij} = |\zeta_i\rangle\langle\zeta_j| \mid i, j = 1, 2, \ldots, N\} \tag{3.1}$$

form an orthonormal basis of $\mathcal{B}_2(\mathfrak{H})$,

$$\langle X_{ij} \mid X_{k\ell} \rangle_2 = \delta_{ik}\delta_{\ell j}.$$

In particular, the vectors

$$\mathbb{P}_i = X_{ii} = |\zeta_i\rangle\langle\zeta_i| \tag{3.2}$$

are one-dimensional projection operators on $\mathfrak{H}$. In what follows I will denote the identity operator on $\mathfrak{H}$ and I_2 that on $\mathcal{B}_2(\mathfrak{H})$.

We identify a special class of linear operators on $\mathcal{B}_2(\mathfrak{H})$, denoted by $A \vee B$, $A, B \in \mathcal{L}(\mathfrak{H})$, which acts on a vector $X \in \mathcal{B}_2(\mathfrak{H})$ in the manner

$$(A \vee B)(X) = AXB^*.$$

Using the scalar product in $\mathcal{B}_2(\mathfrak{H})$, we see that

$$\mathrm{Tr}[X^*(AYB^*)] = \mathrm{Tr}[(A^*XB)^*Y)] \implies (A \vee B)^* = A^* \vee B^*,$$

and since for any $X \in \mathcal{B}_2(\mathfrak{H})$

$$(A_1 \vee B_1)(A_2 \vee B_2)(X) = A_1[(A_2 \vee B_2)(X)]B_1^* = A_1A_2XB_2^*B_1^*,$$

we have

$$(A_1 \vee B_1)(A_2 \vee B_2) = (A_1 A_2) \vee (B_1 B_2). \tag{3.3}$$

There are two special von Neumann algebras which can be built out of these operators. These are

$$\mathfrak{A}_\ell = \{A_\ell = A \vee I \mid A \in \mathcal{L}(\mathfrak{H})\}, \qquad \mathfrak{A}_r = \{A_r = I \vee A \mid A \in \mathcal{L}(\mathfrak{H})\}. \tag{3.4}$$

They are mutual commutants and both are factors:

$$(\mathfrak{A}_\ell)' = \mathfrak{A}_r, \qquad (\mathfrak{A}_r)' = \mathfrak{A}_\ell, \qquad \mathfrak{A}_\ell \cap \mathfrak{A}_r = \mathbb{C} I_2. \tag{3.5}$$

Consider now the operator $J : \mathcal{B}_2(\mathfrak{H}) \longrightarrow \mathcal{B}_2(\mathfrak{H})$, whose action on the vectors X_{ij} in (3.1) is given by

$$JX_{ij} = X_{ji} \Longrightarrow J^2 = I_2 \qquad \text{and} \qquad J(|\phi\rangle\langle\psi|) = |\psi\rangle\langle\phi|, \qquad \forall \phi, \psi \in \mathfrak{H}. \tag{3.6}$$

This operator is antiunitary, and since

$$[J(A \vee I)J]X_{ij} = J(A \vee I)X_{ji} = J(AX_{ji}) = J(A|\zeta_j\rangle\langle\zeta_i|) = |\zeta_i\rangle\langle\zeta_j|A^* = (I \vee A)X_{ij},$$

we immediately get

$$J\mathfrak{A}_\ell J = \mathfrak{A}_r. \tag{3.7}$$

3.1. A KMS state

Let α_i, $i = 1, 2, \ldots, N$ be a sequence of non-zero, positive numbers, satisfying $\sum_{i=1}^{N} \alpha_i = 1$. Let

$$\Phi = \sum_{i=1}^{N} \alpha_i^{\frac{1}{2}} \, \mathbb{P}_i = \sum_{i=1}^{N} \alpha_i^{\frac{1}{2}} \, X_{ii} \in \mathcal{B}_2(\mathfrak{H}). \tag{3.8}$$

We note the following properties of Φ.

(1) Φ defines a vector state φ on the von Neumann algebra $\mathfrak{A}_\ell$. This follows from the fact that for any $A \vee I \in \mathfrak{A}_\ell$, we may define the state φ on $\mathfrak{A}_\ell$ by

$$\langle \varphi; A \vee I \rangle = \langle \Phi \mid (A \vee I)(\Phi) \rangle_2 = \text{Tr}[\Phi^* A \Phi] = \text{Tr}[\rho_\varphi A], \qquad \text{with} \quad \rho_\varphi = \sum_{i=1}^{N} \alpha_i \mathbb{P}_i. \tag{3.9}$$

(2) The state φ is faithful and normal. Normality follows from the last equality in (3.9) and the fact that ρ_φ is a density matrix. To check for faithfulness, note that for any $A \vee I \in \mathfrak{A}_\ell$,

$$\langle \varphi; (A \vee I)^*(A \vee I) \rangle = \text{Tr}[\rho_\varphi A^* A] = \sum_{i=1}^{N} \alpha_i \|A\zeta_i\|^2$$

from which it follows that $\langle \varphi; (A \vee I)^*(A \vee I) \rangle = 0$ if and only if $A = 0$ (since the ζ_i are an orthonormal basis set and the $\alpha_i > 0$), hence if and only if $A \vee I = 0$.

(3) The vector Φ is cyclic and separating for $\mathfrak{A}_\ell$. Indeed, cyclicity follows from the fact that if $X \in \mathcal{B}_2(\mathfrak{H})$ is orthogonal to all $(A \vee I)\Phi$, $A \in \mathcal{L}(\mathfrak{H})$, then

$$\text{Tr}[X^* A \Phi] = \sum_{i=1}^{N} \alpha_i^{\frac{1}{2}} \langle \zeta_i \mid X^* A \zeta_i \rangle = 0, \qquad \forall A \in \mathcal{L}(\mathfrak{H}).$$

Taking $A = X_{k\ell}$, we easily get from the above equality, $\langle \zeta_\ell \mid X^* \zeta_k \rangle = 0$ and since this holds for all k, ℓ, we get $X = 0$. In the same way, Φ is also cyclic for $\mathfrak{A}_r$, hence separating for $\mathfrak{A}_\ell$, i.e. $(A \vee I)\Phi = (B \vee I)\Phi \Longleftrightarrow A \vee I = B \vee I$.

We shall show in the following that the state φ constructed above is indeed a KMS state for a particular choice of α_i.

3.2. Time evolution and modular automorphism

We now construct a time evolution $\alpha_\varphi(t)$, $t \in \mathbb{R}$, on the algebra $\mathfrak{A}_\ell$, using the state φ, with respect to which it has the KMS property, for fixed $\beta > 0$,

$$\langle \varphi; A_\ell \alpha_\varphi(t + \mathrm{i}\beta)[B_\ell] \rangle = \langle \varphi; \alpha_\varphi(t)[B_\ell]A_\ell \rangle, \qquad \forall A_\ell, B_\ell \in \mathfrak{A}_\ell, \tag{3.10}$$

and moreover the function

$$F_{A_\ell, B_\ell}(z) = \langle \varphi; A_\ell \alpha_\varphi(z)[B_\ell] \rangle \tag{3.11}$$

is analytic in the strip $\{\Im(z) \in (0, \beta)\}$ and continuous on its boundaries. We start by defining the operators

$$\mathbf{P}_{ij} = \mathbb{P}_i \vee \mathbb{P}_j, \qquad i, j = 1, 2, \ldots, N, \tag{3.12}$$

where the $\mathbb{P}_i$ are the projection operators on $\mathfrak{H}$ defined in (3.2). Clearly, the $\mathbf{P}_{ij}$ are projection operators on the Hilbert space $\mathcal{B}_2(\mathfrak{H})$.

Using ρ_φ in (3.9) and for a fixed $\beta > 0$, define the operator H_φ as

$$\rho_\varphi = \mathrm{e}^{-\beta H_\varphi} \implies H_\varphi = -\frac{1}{\beta} \sum_{i=1}^{N} (\ln \alpha_i) \, \mathbb{P}_i. \tag{3.13}$$

Next, we define the operators

$$H_\varphi^\ell = H_\varphi \vee I, \qquad H_\varphi^{\mathrm{r}} = I \vee H_\varphi, \qquad \mathbf{H}_\varphi = H_\varphi^\ell - H_\varphi^{\mathrm{r}}. \tag{3.14}$$

Since $\sum_{i=1}^{N} \mathbb{P}_i = I$, we may also write

$$H_\varphi^\ell = -\frac{1}{\beta} \sum_{i,j=1}^{N} \ln \alpha_i \, \mathbf{P}_{ij}, \qquad \text{and} \qquad H_\varphi^{\mathrm{r}} = -\frac{1}{\beta} \sum_{i,j=1}^{N} \ln \alpha_j \, \mathbf{P}_{ij}.$$

Thus,

$$\mathbf{H}_\varphi = -\frac{1}{\beta} \sum_{i,j=1}^{N} \ln \left[\frac{\alpha_i}{\alpha_j} \right] \mathbf{P}_{ij}. \tag{3.15}$$

Using the operator

$$\Delta_\varphi := \sum_{i,j=1}^{N} \left[\frac{\alpha_i}{\alpha_j} \right] \mathbf{P}_{ij} = \mathrm{e}^{-\beta \mathbf{H}_\varphi}, \tag{3.16}$$

we define a time evolution operator on $\mathcal{B}_2(\mathfrak{H})$:

$$\mathrm{e}^{\mathrm{i}\mathbf{H}_\varphi t} = [\Delta_\varphi]^{-\frac{\mathrm{i}}{t}\beta}, \qquad t \in \mathbb{R}, \tag{3.17}$$

and we note that, for any $X \in \mathcal{B}_2(\mathfrak{H})$,

$$\mathrm{e}^{\mathrm{i}\mathbf{H}_\varphi t}(X) = \sum_{i,j=1}^{N} \left[\frac{\alpha_i}{\alpha_j} \right]^{-\frac{\mathrm{i}}{t}\beta} \mathbf{P}_{ij}(X) = \left[\sum_{i=1}^{N} (\alpha_i)^{-\frac{\mathrm{i}}{t}\beta} \mathbb{P}_i \right] \vee \left[\sum_{j=1}^{N} (\alpha_j)^{-\frac{\mathrm{i}}{t}\beta} \mathbb{P}_j(X) \right]$$

$$= \mathrm{e}^{\mathrm{i}H_\varphi t}(X) \, \mathrm{e}^{-\mathrm{i}H_\varphi t},$$

so that

$$\mathrm{e}^{\mathrm{i}\mathbf{H}_\varphi t} = \mathrm{e}^{\mathrm{i}H_\varphi t} \vee \mathrm{e}^{\mathrm{i}H_\varphi t}, \tag{3.18}$$

where H_φ is the operator defined in (3.13). From the definition of the vector Φ in (3.8), it is clear that it commutes with H_φ and hence that it is invariant under this time evolution:

$$\mathrm{e}^{\mathrm{i}\mathbf{H}_\varphi t}(\Phi) = \mathrm{e}^{\mathrm{i}H_\varphi t} \, \Phi \, \mathrm{e}^{-\mathrm{i}H_\varphi t} = \Phi. \tag{3.19}$$

Finally, using $e^{iH_\varphi t}$ we define the time evolution α_φ on the algebra $\mathfrak{A}_\ell$, in the manner (see (2.3))

$$\alpha_\varphi(t)[A_\ell] = e^{iH_\varphi t} A_\ell \, e^{-iH_\varphi t} \qquad \forall A_\ell \in \mathfrak{A}_\ell. \tag{3.20}$$

Writing $A_\ell = A \vee I$, $A \in \mathcal{L}(\mathfrak{H})$, and using the composition law (3.3), we see that

$$e^{iH_\varphi t} \, A_\ell \, e^{-iH_\varphi t} = [e^{iH_\varphi t} A \, e^{-iH_\varphi t}] \vee I, \tag{3.21}$$

so that by virtue of (3.9),

$$\langle \varphi; \alpha_\varphi(t)[A_\ell] \rangle = \mathrm{Tr}[\rho_\varphi \, e^{iH_\varphi t} A \, e^{-iH_\varphi t}] = \langle \varphi; A_\ell \rangle, \tag{3.22}$$

since ρ_φ and H_φ commute. Thus, the state φ is invariant under the time evolution α_φ.

To obtain the KMS condition (3.10), combining (3.20) and (3.21), we first note that, with $A_\ell = A \vee I$, $B_\ell = B \vee I$,

$$A_\ell \alpha_\varphi(t)[B_\ell] = [A e^{iH_\varphi t} B \, e^{-iH_\varphi t}] \vee I.$$

Hence, again using (3.9),

$$F_{A_\ell, B_\ell}(t) = \langle \varphi; A_\ell \alpha_\varphi(t)[B_\ell] \rangle = \mathrm{Tr}[\rho_\varphi A \, e^{iH_\varphi t} B \, e^{-iH_\varphi t}] = \mathrm{Tr}[\rho_\varphi \, e^{-iH_\varphi t} A \, e^{iH_\varphi t} B];$$

the last equality following from the commutativity of ρ_φ and H_φ. Thus, since $\rho_\varphi = e^{-\beta H_\varphi}$,

$$F_{A_\ell, B_\ell}(t + i\beta) = \mathrm{Tr}[\rho_\varphi \, e^{-iH_\varphi t} e^{\beta H_\varphi} A \, e^{iH_\varphi t} \, e^{-\beta H_\varphi} B] = \mathrm{Tr}[e^{-iH_\varphi t} A \, e^{iH_\varphi t} \rho_\varphi B],$$

so that

$$\langle \varphi; A_\ell \alpha_\varphi(t + i\beta)[B_\ell] \rangle = \mathrm{Tr}[\rho_\varphi e^{iH_\varphi t} B \, e^{-iH_\varphi t} A] = \langle \varphi; \alpha_\varphi(t)[B_\ell] A_\ell \rangle,$$

which is the KMS condition.

3.3. The antilinear operator S_φ

We now analyse the antilinear operator $S_\varphi : \mathcal{B}_2(\mathfrak{H}) \longrightarrow \mathcal{B}_2(\mathfrak{H})$, which acts as (see (2.1))

$$S_\varphi(A_\ell \Phi) = A_\ell^* \Phi, \qquad \forall A_\ell \in \mathfrak{A}_\ell. \tag{3.23}$$

Taking $A_\ell = A \vee I$,

$$S_\varphi(A_\ell \Phi) = A_\ell^* \Phi, \qquad \forall A_\ell \in \mathfrak{A}_\ell \quad \Longleftrightarrow \quad S_\varphi(A\Phi) = A^* \Phi, \qquad \forall A \in \mathcal{L}(\mathfrak{H}).$$

Using (3.8) we may write,

$$S_\varphi(A\Phi) = A^* \Phi \quad \Longrightarrow \quad \sum_{i=1}^N \alpha_i^{\frac{1}{2}} S_\varphi(A\mathbb{P}_i) = \sum_{i=1}^N \alpha_i^{\frac{1}{2}} A^* \mathbb{P}_i.$$

Taking $A = X_{k\ell}$ (see (3.1)) and using $X_{k\ell}\mathbb{P}_i = \delta_{\ell i} X_{ki}$, we then get

$$\alpha_\ell^{\frac{1}{2}} S_\varphi(X_{k\ell}) = \alpha_k^{\frac{1}{2}} S_\varphi(X_{\ell k}) \quad \Longrightarrow \quad S_\varphi(X_{k\ell}) = \left[\frac{\alpha_k}{\alpha_\ell}\right]^{\frac{1}{2}} X_{\ell k}. \tag{3.24}$$

Since any $A \in \mathcal{L}(\mathfrak{H})$ can be written as $A = \sum_{i,j=1}^N a_{ij} X_{ij}$, where $a_{ij} = \langle \zeta_i \mid A\zeta_j \rangle$, and furthermore, since $\mathbf{P}_{ij}(X_{k\ell}) = X_{ij}\delta_{ik}\delta_{j\ell}$, we obtain using (3.6) and (3.16)

$$S_\varphi = J[\Delta_\varphi]^{\frac{1}{2}}, \tag{3.25}$$

which in fact also gives the polar decomposition of S_φ.

Thus, we could have obtained, as described in section 2, the time evolution automorphisms $\alpha_\varphi(t)$, $t \in \mathbb{R}$ by analyzing the antilinear operator S_φ, (since $S_\varphi^* S_\varphi = \Delta_\varphi$) directly. Also, from (3.13), (3.16) and (3.18) we see that the modular operator simply defines the *Gibbs state* corresponding to the Hamiltonian $\mathbf{H}_\varphi$.

3.4. The centralizer

As defined in the appendix, the centralizer of $\mathfrak{A}_\ell$, with respect to the state φ, is the von Neumann algebra:

$$\mathfrak{M}_\varphi = \{B_\ell \in \mathfrak{A}_\ell \mid \langle \varphi; [B_\ell, A_\ell] \rangle = 0, \forall A_\ell \in \mathfrak{A}_\ell\}. \tag{3.26}$$

Let us determine this von Neumann algebra. Writing $A_\ell = A \vee I$, $B_\ell = B \vee I$, the commutator, $[B_\ell, A_\ell] = (AB - BA) \vee I$. Hence, by (3.9),

$$\langle \varphi; [B_\ell, A_\ell] \rangle = \text{Tr}[\rho_\varphi (AB - BA)].$$

Thus, in order for the above expression to vanish, we must have,

$$\sum_{i=1}^{N} \alpha_i \langle \zeta_i \mid AB\zeta_i \rangle = \sum_{i=1}^{N} \alpha_i \langle \zeta_i \mid BA\zeta_i \rangle, \qquad \forall A \in \mathcal{L}(\mathfrak{H}).$$

Taking $A = |\zeta_k\rangle\langle\zeta_\ell|$, this gives

$$\alpha_k \langle \zeta_\ell \mid B\zeta_k \rangle = \alpha_\ell \langle \zeta_\ell \mid B\zeta_k \rangle, \qquad \forall k, \ell = 1, 2, \ldots, N,$$

and since in general, $\alpha_k \neq \alpha_\ell$, this implies that $\langle \zeta_\ell \mid B\zeta_k \rangle = 0$, whenever $k \neq \ell$. Thus, B is of the general form $B = \sum_{i=1}^{N} b_i \mathbb{P}_i$, $b_i \in \mathbb{C}$. In other words, the centralizer $\mathfrak{M}_\varphi$ is generated by the projectors $\mathbb{P}_i^\ell = \mathbb{P}_i \vee I$, $i = 1, 2, \ldots, N$, which are *minimal* (i.e. they do not contain projectors onto smaller subspaces) in $\mathfrak{A}_\ell$. Alternatively, we may write $\mathfrak{M}_\varphi = \{H_\varphi^\ell\}''$, where H_φ^ℓ is the Hamiltonian defined in (3.14), so that it is an *atomic, commutative* von Neumann algebra.

4. Application to Landau levels

We now show how the above setup, based on $\mathcal{B}_2(\mathfrak{H})$, can be applied to a specific physical situation, namely to the case of an electron subject to a constant magnetic field, as discussed in [2].

In that case, $\mathfrak{H} = L^2(\mathbb{R})$ and the mapping $\mathcal{W} : \mathcal{B}_2(\mathfrak{H}) \longrightarrow L^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y)$, with

$$(\mathcal{W}X)(x, y) = \frac{1}{(2\pi)^{\frac{1}{2}}} \text{Tr}[U(x, y)^* X], \qquad \text{where} \quad U(x, y) = \mathrm{e}^{-\mathrm{i}(xQ+yP)}, \tag{4.1}$$

Q, P being the usual position and momentum operators ($[Q, P] = \mathrm{i}I$), transfers the whole modular structure unitarily to the Hilbert space $\widetilde{\mathfrak{H}} = L^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y)$. The mapping $\mathcal{W}$ is often referred to as the *Wigner transform* in the physical literature.

To work this out in some detail, we start by constructing the Hamiltonian H_φ (see (3.13)), using the oscillator Hamiltonian $H_{\text{osc}} = \frac{1}{2}(P^2 + Q^2)$ on $\mathfrak{H}$. Let us choose the orthonormal basis set of vectors ζ_n, $n = 0, 1, 2, \ldots \infty$ to be the eigenvectors of H_{osc}:

$$H_{\text{osc}}\zeta_n = \left(n + \frac{1}{2}\right)\zeta_n. \tag{4.2}$$

As is well known, the ζ_n are the Hermite functions

$$\zeta_n(x) = \frac{1}{\pi^{\frac{1}{4}}} \frac{1}{\sqrt{2^n\,n!}} \, \mathrm{e}^{-\frac{x^2}{2}} h_n(x); \tag{4.3}$$

the h_n being the Hermite polynomials, obtainable as

$$h_n(x) = (-1)^n \, \mathrm{e}^{x^2} \partial_x^n \, \mathrm{e}^{-x^2}. \tag{4.4}$$

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Consider now the operator $e^{-\beta H_{\mathrm{osc}}}$, for some fixed $\beta > 0$. We have

$$e^{-\beta H_{\mathrm{osc}}} = \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\beta} \mathbb{P}_n \qquad \text{and} \qquad \mathrm{Tr}[e^{-\beta H_{\mathrm{osc}}}] = \frac{e^{-\frac{\beta}{2}}}{1 - e^{-\beta}}.$$

Thus, we take

$$\rho_\varphi = \frac{e^{-\beta H_{\mathrm{osc}}}}{\mathrm{Tr}[e^{-\beta H_\varphi}]} = (1 - e^{-\beta}) \sum_{n=0}^{\infty} e^{-n\beta} \mathbb{P}_n \qquad \text{and} \qquad \Phi = [1 - e^{-\beta}]^{\frac{1}{2}} \sum_{n=0}^{\infty} e^{-\frac{n}{2}\beta} \mathbb{P}_n. \quad (4.5)$$

Following (3.9) and (3.13), we write

$$\rho_\varphi = \sum_{n=0}^{\infty} \alpha_n \mathbb{P}_n, \qquad \alpha_n = (1 - e^{-\beta}) e^{-n\beta},$$

and

$$H_\varphi = -\frac{1}{\beta} \sum_{n=0}^{\infty} \ln\left[(1 - e^{-\beta}) e^{-n\beta}\right] \mathbb{P}_n = \sum_{n=0}^{\infty} \left[n - \frac{\ln(1 - e^{-\beta})}{\beta}\right] \mathbb{P}_n$$

$$= H_{\mathrm{osc}} - \left[\frac{1}{2} + \frac{\ln(1 - e^{-\beta})}{\beta}\right] I, \qquad (4.6)$$

which is the Hamiltonian giving the time evolution $\alpha_\varphi(t)$, with respect to which the above ρ_φ defines the KMS state φ. Since the difference between H_φ and H_{osc} is just a constant, we shall identify these two Hamiltonians in the following.

As stated earlier, the dynamical model that we consider is that of a single electron of unit charge, placed in the xy-plane and subjected to a constant magnetic field, pointing along the *positive z-direction*. The classical Hamiltonian of the system, in some convenient units, is

$$H_{\mathrm{elec}} = \frac{1}{2} (\vec{p} - \vec{A})^2 = \frac{1}{2} \left(p_x + \frac{y}{2}\right)^2 + \frac{1}{2} \left(p_y - \frac{x}{2}\right)^2, \qquad (4.7)$$

where we have chosen the magnetic vector potential to be $\vec{A}^\uparrow := \vec{A} = \frac{1}{2}(-y, x, 0)$ (so that the magnetic field $\vec{B} = \nabla \times \vec{A}^\uparrow = (0, 0, 1)$).

Next, on $\widetilde{\mathfrak{H}} = L^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y)$, we introduce the quantized observables

$$p_x + \frac{y}{2} \longrightarrow Q_- = -\mathrm{i}\frac{\partial}{\partial x} + \frac{y}{2}, \qquad p_y - \frac{x}{2} \longrightarrow P_- = -\mathrm{i}\frac{\partial}{\partial y} - \frac{x}{2}, \qquad (4.8)$$

which satisfy $[Q_-, P_-] = \mathrm{i}I_{\widetilde{\mathfrak{H}}}$ and in terms of which the quantum Hamiltonian corresponding to H_{elec} becomes

$$H^\uparrow = \frac{1}{2}\left(P_-^2 + Q_-^2\right). \qquad (4.9)$$

This is the same as the oscillator Hamiltonian in one dimension, H_{osc}, given above (and hence the same as H_φ in (4.6), with our convention of identifying these two). The eigenvalues of this Hamiltonian are then $E_\ell = \left(\ell + \frac{1}{2}\right)$, $\ell = 0, 1, 2, \ldots \infty$. However, this time each level is infinitely degenerate, and we will denote the corresponding normalized eigenvectors by $\Psi_{n\ell}$, with $\ell = 0, 1, 2, \ldots, \infty$, indexing the energy level and $n = 0, 1, 2, \ldots, \infty$, the degeneracy at each level. If the magnetic field were aligned along the *negative z-axis* (with $\vec{A}^\downarrow = \frac{1}{2}(y, -x, 0)$ and $\vec{B} = \nabla \times \vec{A}^\downarrow = (0, 0, -1)$), the corresponding quantum Hamiltonian would have been

$$H^\downarrow = \frac{1}{2}\left(P_+^2 + Q_+^2\right), \qquad (4.10)$$

with

$$Q_+ = -\mathrm{i}\frac{\partial}{\partial y} + \frac{x}{2}, \qquad P_+ = -\mathrm{i}\frac{\partial}{\partial x} - \frac{y}{2}, \qquad (4.11)$$

and $[Q_+, P_+] = \mathrm{i} I_{\widetilde{\mathfrak{H}}}$. The two sets of operators $\{Q_\pm, P_\pm\}$, mutually commute

$$[Q_+, Q_-] = [P_+, Q_-] = [Q_+, P_-] = [P_+, P_-] = 0. \tag{4.12}$$

Thus, $[H^\downarrow, H^\uparrow] = 0$ and the eigenvectors $\Psi_{n\ell}$ of $H^\uparrow$ can be chosen so that they are also the eigenvectors of $H^\downarrow$ in the manner

$$H^\downarrow \Psi_{n\ell} = \left(n + \tfrac{1}{2}\right) \Psi_{n\ell}, \qquad H^\uparrow \Psi_{n\ell} = \left(\ell + \tfrac{1}{2}\right) \Psi_{n\ell}, \tag{4.13}$$

so that $H^\downarrow$ lifts the degeneracy of $H^\uparrow$ and vice versa. In what follows, we shall assume that this is the case.

Then, it is well known (see, for example, [1]) that the map $\mathcal{W}$ in (4.1) is unitary and straightforward computations (see, for example [2]) yield,

$$\mathcal{W} \begin{pmatrix} Q \vee I_{\mathfrak{H}} \\ P \vee I_{\mathfrak{H}} \end{pmatrix} \mathcal{W}^{-1} = \begin{pmatrix} Q_+ \\ P_+ \end{pmatrix}, \qquad \mathcal{W} \begin{pmatrix} I_{\mathfrak{H}} \vee Q \\ I_{\mathfrak{H}} \vee P \end{pmatrix} \mathcal{W}^{-1} = \begin{pmatrix} Q_- \\ P_- \end{pmatrix}, \tag{4.14}$$

and

$$\mathcal{W} \begin{pmatrix} H_{\mathrm{osc}} \vee I_{\mathfrak{H}} \\ I_{\mathfrak{H}} \vee H_{\mathrm{osc}} \end{pmatrix} \mathcal{W}^{-1} = \begin{pmatrix} H^\downarrow \\ H^\uparrow \end{pmatrix}, \qquad \mathcal{W} X_{n\ell} = \Psi_{n\ell}, \tag{4.15}$$

where the $X_{n\ell}$ are the basis vectors defined in (3.1) and the $\Psi_{n\ell}$ are the normalized eigenvectors defined in (4.13). This also means that these latter vectors form a basis of $\widetilde{\mathfrak{H}} = L^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y)$. Finally, note that the two sets of operators, $\{Q_+, P_+\}$ and $\{Q_-, P_-\}$, generate (see appendix) the two von Neumann algebras $\mathfrak{A}_+$ and $\mathfrak{A}_-$, respectively, with $\mathcal{W}\mathfrak{A}_\ell \mathcal{W}^{-1} = \mathfrak{A}_+$ and $\mathcal{W}\mathfrak{A}_r \mathcal{W}^{-1} = \mathfrak{A}_-$. Thus physically, the two commuting algebras correspond to the two directions of the magnetic field. The KMS state $\Psi = \mathcal{W}\Phi$, with Φ given by (4.5) is just the *Gibbs equilibrium state* for this physical system. Let us note that the parameter β, introduced in the definition of the KMS state and representing an inverse temperature, has not been fixed in this discussion. In an experimental setup, Landau levels are observed for very strong fields and low temperatures, i.e. for $\beta \gg 1$.

Remark: The problem of *lifting the degeneracy* is a major point in the analysis and understanding of the quantum Hall effect: it is exactly this infinite degeneracy for the single electron Hamiltonian which allows for the existence of different many-body wavefunctions. It is also known that different electron densities (measured by the so-called filling factor ν) correspond to physically (and not only mathematically) different wavefunctions, describing a Wigner crystal for small ν or an incompressible fluid, for larger ν, [5]. Mathematically, it is clear from the analysis that if the magnetic field is pointing upwards, so that the Hamiltonian is given by (4.9), an additional term in the Hamiltonian depending on the other two operators, P_+, Q_+, such as might arise from a crossed electric field or confining potential (see, e.g. [7]), would lift the degeneracy of the levels. Physically, the degeneracy of the Landau levels is explained by the impossibility of quantum mechanically fixing the origin of the centre of the circular orbits of the electron.

5. A second representation

It is interesting to pursue this example a bit further by transforming to complex coordinates, which will essentially reduce the action of the operator J to one of the complex conjugation. The possibility of having this other representation is a reflection of the fact that there is more than one possible way to represent the two commuting von Neumann algebras $\mathfrak{A}_\pm$. As before, let us consider the electron in a uniform magnetic field oriented in the positive z-direction, with vector potential $\vec{A}^\uparrow = \tfrac{1}{2}(-y, x, 0)$ and magnetic field $\vec{B} = \nabla \times \vec{A}^\uparrow = (0, 0, 1))$. The

classical Hamiltonian is now given by $H^\uparrow = \frac{1}{2}(\vec{p} - \vec{A}^\uparrow)^2$. There are several possible ways to write this Hamiltonian, which are more convenient than using the coordinates x, y and z. One such representation was used in section 4 and we indicate below a second possibility. Note that the quantized Hamiltonian may be split into a free part H_0 and an interaction or angular momentum part $H_{int}^\uparrow$

$$\begin{cases} H^\uparrow = H_0 + H_{int}^\uparrow, \\ H_0 = H_{0,x} + H_{0,y} = \frac{1}{2}\left(\widehat{p}_x^2 + \frac{\widehat{x}^2}{4}\right) + \frac{1}{2}\left(\widehat{p}_y^2 + \frac{\widehat{y}^2}{4}\right), \\ H_{int}^\uparrow = -\frac{1}{2}(\widehat{x}\widehat{p}_y - \widehat{y}\widehat{p}_x) = -\widehat{l}_z. \end{cases} \tag{5.1}$$

with the usual definitions of $\widehat{x}$, $\widehat{p}_x$, etc. Of course, $[\widehat{x}, \widehat{p}_x] = [\widehat{y}, \widehat{p}_y] = iI_{\widetilde{\mathfrak{H}}}$, while all the other commutators are zero. Introducing the corresponding annihilation operators,

$$a_x = \frac{1}{\sqrt{2}}[\widehat{x} + i\widehat{p}_x], \qquad a_y = \frac{1}{\sqrt{2}}[\widehat{y} + i\widehat{p}_y], \tag{5.2}$$

and their adjoints

$$a_x^* = \frac{1}{\sqrt{2}}[\widehat{x} - i\widehat{p}_x], \qquad a_y^* = \frac{1}{\sqrt{2}}[\widehat{y} - i\widehat{p}_y], \tag{5.3}$$

which satisfy the canonical commutation rules $[a_x, a_x^*] = [a_y, a_y^*] = I_{\widetilde{\mathfrak{H}}}$; while all the other commutators are zero, the Hamiltonian $H^\uparrow$ can be written as $H^\uparrow = H_0 + H_{int}^\uparrow$, with $H_0 = (a_x^* a_x + a_y^* a_y + I_{\widetilde{\mathfrak{H}}})$, $H_{int}^\uparrow = -i(a_x a_y^* - a_y a_x^*)$. $H^\uparrow$ does not appear to be diagonal even in this form, so that another *change of variables* is required.

Using the operators $Q_\pm$, $P_\pm$, given in (4.8) and (4.11), let us define

$$\begin{aligned} A_+ &= \frac{1}{\sqrt{2}}(Q_+ + iP_+) = \frac{3}{4}(a_x - ia_y) - \frac{1}{4}(a_x^* + ia_y^*), \\ A_+^* &= \frac{1}{\sqrt{2}}(Q_+ - iP_+) = \frac{3}{4}(a_x^* + ia_y^*) - \frac{1}{4}(a_x - ia_y), \\ A_- &= \frac{1}{\sqrt{2}}(iQ_- - P_-) = \frac{3}{4}(a_x + ia_y) - \frac{1}{4}(a_x^* + ia_y^*), \\ A_-^* &= \frac{1}{\sqrt{2}}(-iQ_- - P_-) = \frac{3}{4}(a_x^* - ia_y^*) - \frac{1}{4}(a_x - ia_y). \end{aligned} \tag{5.4}$$

These satisfy the commutation relations

$$[A_\pm, A_\pm^*] = 1, \tag{5.5}$$

with all other commutators being zero. In terms of these, we may write the two Hamiltonians as (see (4.9) and (4.10))

$$H^\uparrow = N_- + \tfrac{1}{2}I_{\widetilde{\mathfrak{H}}}, \qquad H^\downarrow = N_+ + \tfrac{1}{2}I_{\widetilde{\mathfrak{H}}}, \qquad \text{with} \quad N_\pm = A_\pm^* A_\pm. \tag{5.6}$$

Furthermore,

$$H_0 = \tfrac{1}{2}(N_+ + N_- + 1) \qquad \text{and} \qquad H_{int}^\uparrow = -\tfrac{1}{2}(N_+ - N_-), \qquad H_{int}^\downarrow = \tfrac{1}{2}(N_+ - N_-). \tag{5.7}$$

The eigenstates of $H^\uparrow$ are now easily written down. Let Ψ_{00} be such that $A_- \Psi_{00} = A_+ \Psi_{00} = 0$. Then we define

$$\Psi_{n\ell} := \frac{1}{\sqrt{n!\ell!}}(A_+^*)^n (A_-^*)^\ell \Psi_{00}, \tag{5.8}$$

where $n, \ell = 0, 1, 2, \ldots$. All the relevant operators are now diagonal in this basis: $N_+ \Psi_{n\ell} = n\Psi_{n\ell}$, $N_- \Psi_{n\ell} = \ell\Psi_{n\ell}$, $H_0 \Psi_{n\ell} = \frac{1}{2}(n + \ell + 1)\Psi_{n\ell}$ and $H_{int}^\uparrow \Psi_{n\ell} = \frac{1}{2}(n - \ell)\Psi_{n\ell}$. Hence,

$$H^\uparrow \Psi_{n\ell} = \left(\ell + \tfrac{1}{2}\right)\Psi_{n\ell}. \tag{5.9}$$

This means that each level ℓ is infinitely degenerate, with n being the degeneracy index. Again, this degeneracy can be lifted in a physically interesting way namely, by considering the *reflected* magnetic field with vector potential $\vec{A}^{\downarrow} = \frac{1}{2}(y, -x, 0)$ as in section 4, with the magnetic field directed along the negative z-direction. The same electron considered above is now described by the other Hamiltonian, $H^{\downarrow}$, which can be written as

$$\begin{cases} H^{\downarrow} = \frac{1}{2}(\vec{p} - \vec{A}^{\downarrow})^2 = H_0 + H_{\text{int}}^{\downarrow}, \\ H_{\text{int}}^{\downarrow} = -H_{\text{int}}^{\uparrow}. \end{cases} \tag{5.10}$$

Thus, since $H^{\downarrow}$ can also be written as in (5.6), its eigenstates are again the same $\Psi_{n\ell}$ given in (5.8). (Recall that $[H^{\uparrow}, H^{\downarrow}] = 0$, so that they can be simultaneously diagonalized.) This also means that, as in (4.15), $\mathcal{W}X_{n\ell} = \Psi_{n\ell}$ and the closure of the linear span of the $\Psi_{n\ell}$'s is the Hilbert space $\widetilde{\mathfrak{H}} = L^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y)$.

However, this is not the end of the story. Indeed, introducing the complex variable $z = \frac{1}{\sqrt{2}}(x - \mathrm{i}y)$ and its associated derivative $\partial_z = \frac{1}{\sqrt{2}}(\partial_x + \mathrm{i}\partial_y)$, the operators A_+ and A_- can be written as

$$A_- = \tfrac{1}{2}\bar{z} + \partial_z, \qquad A_+ = \tfrac{1}{2}z + \partial_{\bar{z}}, \tag{5.11}$$

and their adjoints as

$$A_-^* = \tfrac{1}{2}z - \partial_{\bar{z}}, \qquad A_+^* = \tfrac{1}{2}\bar{z} - \partial_z. \tag{5.12}$$

In this $(\bar{z}, z)$-representation the ground state $\Psi_{00}(\bar{z}, z)$ is the solution of the equations $A_+\Psi_{00}(\bar{z}, z) = A_-\Psi_{00}(\bar{z}, z) = 0$, so that $\Psi_{00}(\bar{z}, z) = \sqrt{\frac{1}{2\pi}}\,\mathrm{e}^{-\frac{1}{2}|z|^2}$. We conclude from (4.15) and (5.8) that

$$\Psi_{n\ell}(\bar{z}, z) = (\mathcal{W}X_{n\ell})(\bar{z}, z) = \frac{1}{\sqrt{n!\ell!}}\left(\frac{1}{2}\bar{z} - \partial_z\right)^n \left(\frac{1}{2}z - \partial_{\bar{z}}\right)^\ell \Psi_{00}(z, \bar{z}). \tag{5.13}$$

When using this representation, we shall write our Hilbert space as $\widetilde{\mathfrak{H}} = L^2(\mathbb{C}, \frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{\mathrm{i}})$ and then it is useful to make the further unitary transformation

$$\mathcal{V} : L^2\left(\mathbb{C}, \frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{\mathrm{i}}\right) \longrightarrow L^2(\mathbb{C}, \mathrm{d}\nu(\bar{z}, z)) \qquad \text{where} \quad \mathrm{d}\nu(\bar{z}, z) = \frac{\mathrm{e}^{-|z|^2}}{2\pi}\,\frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{\mathrm{i}}, \tag{5.14}$$

to the more convenient Hilbert space $L^2(\mathbb{C}, \mathrm{d}\nu(\bar{z}, z))$, using the mapping

$$(\mathcal{V}\Psi)(\bar{z}, z) = \sqrt{2\pi}\,\mathrm{e}^{\frac{|z|^2}{2}}\,\Psi(\bar{z}, z), \tag{5.15}$$

and to rewrite all the operators in question on this new space. Note that this space contains the two subspaces $\mathfrak{H}_{\text{hol}}$ and $\mathfrak{H}_{\text{a-hol}}$, of holomorphic and antiholomorphic functions, respectively. Both these subspaces contain the constant unit vector, $H_{00}(\bar{z}, z) = 1, \forall(\bar{z}, z)$. Apart from this one vector, all other vectors in the complementary subspaces of $\mathfrak{H}_{\text{hol}}$ and $\mathfrak{H}_{\text{a-hol}}$ are mutually orthogonal. Since

$$\frac{\partial}{\partial z}\left[\Psi(\bar{z}, z)\,\mathrm{e}^{\frac{|z|^2}{2}}\right] = \left[\frac{\partial}{\partial z}\Psi(\bar{z}, z) + \bar{z}\Psi(\bar{z}, z)\right]\mathrm{e}^{\frac{|z|^2}{2}},$$

we immediately find that

$$\begin{aligned} \mathcal{A}_- &:= \mathcal{V}A_-\mathcal{V}^{-1} = \partial_z, & \mathcal{A}_+ &:= \mathcal{V}A_+\mathcal{V}^{-1} = \partial_{\bar{z}}, \\ \mathcal{A}_-^* &:= \mathcal{V}A_-^*\mathcal{V}^{-1} = z - \partial_{\bar{z}}, & \mathcal{A}_+^* &:= \mathcal{V}A_+^*\mathcal{V}^{-1} = \bar{z} - \partial_z, \end{aligned} \tag{5.16}$$

Furthermore, in this representation we have the number operators

$$\begin{aligned} \mathcal{N}_+ &:= \mathcal{V}N_+\mathcal{V}^{-1} = \mathcal{A}_+^*\mathcal{A}_+ = -\partial_z\partial_{\bar{z}} + \bar{z}\partial_{\bar{z}} \\ \mathcal{N}_- &:= \mathcal{V}N_-\mathcal{V}^{-1} = \mathcal{A}_-^*\mathcal{A}_- = -\partial_z\partial_{\bar{z}} + z\partial_z. \end{aligned} \tag{5.17}$$

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Writing $H_{n\ell} = \mathcal{V}\Psi_{n\ell}$, for the transformed basis vectors (5.13), we have

$$H_{n\ell}(\bar{z}, z) = \frac{1}{\sqrt{n!\ell!}}(\bar{z} - \partial_z)^n (z - \partial_{\bar{z}})^\ell H_{00}(\bar{z}, z) = \frac{1}{\sqrt{n!\ell!}}(\mathcal{A}_+^*)^n (\mathcal{A}_-^*)^\ell H_{00}(\bar{z}, z),$$

$$= \frac{1}{\sqrt{n!\ell!}}(\bar{z} - \partial_z)^n (z^\ell) = \frac{1}{\sqrt{n!\ell!}}(z - \partial_{\bar{z}})^\ell (\bar{z})^n. \tag{5.18}$$

Also,

$$H_{n0}(\bar{z}, z) = \frac{\bar{z}^n}{\sqrt{n!}} \qquad \text{and} \qquad H_{0\ell}(\bar{z}, z) = \frac{z^\ell}{\sqrt{\ell!}}, \tag{5.19}$$

so that $\mathfrak{H}_{\text{a-hol}}$ is spanned by the vectors H_{n0}, $n = 0, 1, 2, \ldots$, and the space $\mathfrak{H}_{\text{hol}}$ by the vectors $H_{0\ell}$, $\ell = 0, 1, 2, \ldots$.

The vectors $H_{n\ell}$ are joint eigenstates of the number operators:

$$\mathcal{N}_+ H_{n\ell} = \ell H_{n\ell}, \qquad \mathcal{N}_- H_{n\ell} = n H_{n\ell}. \tag{5.20}$$

Moreover, writing

$$\mathcal{H}^\uparrow = \mathcal{V}H^\uparrow \mathcal{V}^{-1}, \qquad \mathcal{H}^\downarrow = \mathcal{V}H^\downarrow \mathcal{V}^{-1},$$

$$\mathcal{H}_0 = \mathcal{V}H_0 \mathcal{V}^{-1}, \qquad \mathcal{H}_{\text{int}}^{\uparrow,\downarrow} = \mathcal{V}H_{\text{int}}^{\uparrow,\downarrow} \mathcal{V}^{-1}, \tag{5.21}$$

for the two Hamiltonians, we clearly have (see (5.9))

$$\mathcal{H}^\uparrow H_{n\ell} = \left(\ell + \tfrac{1}{2}\right) H_{n\ell}, \qquad \mathcal{H}^\downarrow H_{n\ell} = \left(n + \tfrac{1}{2}\right) H_{n\ell}. \tag{5.22}$$

The functions $h_{n,k}(\bar{z}, z) = \sqrt{n!k!}H_{nk}(\bar{z}, z)$ are just the *complex Hermite polynomials* [10, 11], also obtainable as

$$h_{n,k}(\bar{z}, z) = (-1)^{n+k} e^{|z|^2} \partial_z^n \partial_{\bar{z}}^k e^{-|z|^2}. \tag{5.23}$$

Explicitly, the $h_{n,k}$ are given by

$$h_{n,k}(\bar{z}, z) = n! \, k! \sum_{j=0}^{n \curlyvee k} \frac{(-1)^j}{j!} \frac{(\bar{z})^{n-j}}{(n-j)!} \frac{z^{k-j}}{(k-j)!}, \tag{5.24}$$

where $n \curlyvee k$ denotes the smaller of the two numbers n and k. In particular,

$$h_{0,k}(\bar{z}, z) = z^k \qquad \text{and} \qquad h_{n,0}(\bar{z}, z) = \bar{z}^n. \tag{5.25}$$

One also has the useful series expansion

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{v^n \bar{u}^k}{n!k!} h_{n,k}(\bar{z}, z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{v^n \bar{u}^k}{\sqrt{n!k!}} H_{nk}(\bar{z}, z) = e^{\bar{u}z + v\bar{z} - \bar{u}v}. \tag{5.26}$$

Furthermore,

$$h_{n,k}(\bar{z}, z) = h_{k,n}(z, \bar{z}) \qquad \text{and} \qquad h_{n,k}(\bar{z}, z) = ((\mathcal{A}_+^*)^n h_{0,k})(\bar{z}, z). \tag{5.27}$$

They also satisfy the recursion relations,

$$h_{n+1,k}(\bar{z}, z) = \bar{z} \, h_{n,k}(\bar{z}, z) - k \, h_{n,k-1}(\bar{z}, z)$$

$$h_{n,k+1}(\bar{z}, z) = z \, h_{n,k}(\bar{z}, z) - n \, h_{n-1,k}(\bar{z}, z), \tag{5.28}$$

from which we further obtain

$$\bar{z} h_{n,n+1}(\bar{z}, z) = z \, h_{n+1,n}(\bar{z}, z)$$

$$(k - m) \, h_{m,k}(\bar{z}, z) = \bar{z} \, h_{m,k+1}(\bar{z}, z) - z \, h_{m+1,k}(\bar{z}, z). \tag{5.29}$$

If we formally take z to be real in (5.23), the complex Hermite polynomials $h_{n,k}(\bar{z}, z)$ become the well-known real Hermite polynomials $h_{n+k}(x)$ (see (4.4)):

$$h_n(x) = (-1)^n e^{x^2} \partial_x^n e^{-x^2},$$

which satisfy the recursion relations

$$xh_n(x) = nh_{n-1}(x) + \tfrac{1}{2}h_{n+1}(x). \tag{5.30}$$

6. Some physical considerations

The composite map

$$\mathcal{U} := \mathcal{V} \circ \mathcal{W} : \mathcal{B}_2(\mathfrak{H}) \longrightarrow L^2(\mathbb{C}, d\nu(\overline{z}, z)), \qquad \text{with} \quad \mathcal{U}X_{n\ell} = H_{n\ell}, \tag{6.1}$$

transforms the modular conjugation map J in (3.6) to $\mathcal{J} = \mathcal{U}J\mathcal{U}^{-1}$ which basically acts by complex conjugation (see (5.16) and (5.27))

$$(\mathcal{J}H_{n\ell})(\overline{z}, z) = H_{\ell n}(\overline{z}, z) = \overline{H_{n\ell}(z, \overline{z})} \qquad \text{and} \qquad \mathcal{J}\mathcal{A}_{\pm}\mathcal{J}^{-1} = \mathcal{A}_{\mp} = \overline{\mathcal{A}_{\pm}}, \tag{6.2}$$

etc. (The overline indicates the complex conjugation of the variables appearing in the definitions of the operators.). Similarly, the two mutually commuting algebras, $\mathfrak{U}_+ = \mathcal{U}\,\mathfrak{A}_+\mathcal{U}^{-1}$ and $\mathfrak{U}_- = \mathcal{U}\,\mathfrak{A}_-\mathcal{U}^{-1}$, generated by the two sets of operators $\{\mathcal{A}_+, \mathcal{A}_+^*\}$ and $\{\mathcal{A}_-, \mathcal{A}_-^*\}$, respectively, are transformed into each other by $\mathcal{J}$, both algebras being factors. Additionally,

$$\mathcal{J}\mathcal{H}_0\mathcal{J} = \mathcal{H}_0, \qquad \mathcal{J}\mathcal{H}_{\text{int}}^{\uparrow}\mathcal{J} = \mathcal{H}_{\text{int}}^{\downarrow} = -\mathcal{H}_{\text{int}}^{\uparrow} \quad \Rightarrow \quad \mathcal{J}\mathcal{H}^{\uparrow}\mathcal{J} = \mathcal{H}^{\downarrow}. \tag{6.3}$$

The KMS state on the algebra $\mathfrak{U}_+$, which is a vector state, is given by the vector (see (3.8) and (4.5))

$$\boldsymbol{\mathcal{X}} = \mathcal{U}\boldsymbol{\Phi} = (1 - e^{-\beta})^{\frac{1}{2}} \sum_{n=0}^{\infty} e^{-\frac{n\beta}{2}} H_{nn} \in L^2(\mathbb{C}, d\nu(\overline{z}, z)), \qquad \mathcal{J}\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{X}}. \tag{6.4}$$

The Hamiltonian

$$\boldsymbol{\mathcal{H}}_{\varphi} = \mathcal{H}^{\uparrow} - \mathcal{H}^{\downarrow} = 2\mathcal{H}_{\text{int}}^{\uparrow} = -2(\mathcal{N}_+ - \mathcal{N}_-), \tag{6.5}$$

then gives the modular operator

$$\Delta_{\varphi} = \exp[-\beta\boldsymbol{\mathcal{H}}_{\varphi}] = \sum_{n,\ell=0}^{\infty} e^{-\beta(n-\ell)} |H_{n\ell}\rangle\langle H_{n\ell}|, \tag{6.6}$$

and the one-parameter automorphism group

$$\Delta_{\varphi}^{-\frac{i}{t}\beta} = \exp[i\boldsymbol{\mathcal{H}}_{\varphi}t] = \exp[2i\mathcal{H}_{\text{int}}^{\uparrow}t], t \in \mathbb{R}, \qquad \Delta_{\varphi}^{-\frac{i}{t}\beta}\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{X}}, \tag{6.7}$$

which stabilizes $\boldsymbol{\mathcal{X}}$. In other words, the modular automorphism is basically the time evolution generated by the interaction Hamiltonian. One also verifies that

$$\Delta_{\varphi}^{-\frac{i}{t}\beta}\mathfrak{U}_+\Delta_{\varphi}^{\frac{i}{t}\beta} = \mathfrak{U}_+. \tag{6.8}$$

The conclusion is therefore the following: *the map $\mathcal{J}$ in (6.2) is, at the same time*,

- the modular map of the Tomita–Takesaki theory;
- the complex conjugation map;
- the map which reverses the uniform magnetic field, from $\vec{B}$ to $-\vec{B}$, thus transforming $\mathcal{H}^{\uparrow}$ to $\mathcal{H}^{\downarrow}$, while leaving $\mathcal{H}_0$ unaffected;
- the operator interchanging the two mutually commuting von Neumann algebras $\mathfrak{U}_{\pm}$, these latter defining, therefore, the experimental setups corresponding to the two directions of the magnetic field;
- an intertwining operator in the sense of [12], see below.

Let us consider this last claim in a bit more detail, following [6, 12] and references therein. The main result on this topic is the following: suppose we have two Hamiltonians, H_1 and H_2, which are related by an intertwining operator X in the following way: $XH_1 = H_2X$. Then, the knowledge of the eigensystem of, say, H_1 essentially fixes the eigensystem of H_2. Indeed we have [12] that if $\phi_n^{(1)}$ is an eigenstate of H_1 with eigenvalue E_n, $H_1\phi_n^{(1)} = E_n\phi_n^{(1)}$, then $X\phi_n^{(1)}$ is either zero or is an eigenstate of H_2 with the same eigenvalue: $H_2(X\phi_n^{(1)}) = E_n(X\phi_n^{(1)})$. This

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is just a consequence of the intertwining relation. In [6] this approach has been generalized, proposing a procedure to build up H_2 from H_1 and from a certain operator which plays the role of the X above.

Writing $\mathcal{J}\mathcal{H}^{\uparrow}\mathcal{J} = \mathcal{H}^{\downarrow}$ as $\mathcal{J}\mathcal{H}^{\uparrow} = \mathcal{H}^{\downarrow}\mathcal{J}$, we see that $\mathcal{J}$ is an intertwining operator between $\mathcal{H}^{\downarrow}$ and $\mathcal{H}^{\uparrow}$. Hence, if Φ is an eigenstate of $\mathcal{H}^{\uparrow}$ with eigenvalue E, then $\mathcal{J}\Phi$ is either zero or is an eigenvector of $\mathcal{H}^{\downarrow}$ with the same eigenvalue. This is exactly what is explicitly expressed by equations (5.22) and (6.2) above.

7. Bi-coherent states and conjugate coherent states

There are several natural families of coherent states associated with the Hamiltonians $\mathcal{H}^{\uparrow}, \mathcal{H}^{\downarrow}$ and $\mathcal{H}_{\varphi} = \mathcal{H}^{\uparrow} - \mathcal{H}^{\downarrow}$. These were constructed in a different context in [2]. Here we look at them again using the complex Hermite polynomials and the modular conjugation. Using the expansion in (5.26), we define the (non-normalized) *bi-coherent states*:

$$\eta_{\overline{u},v}^{\mathrm{bcs}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{v^n \overline{u}^k}{\sqrt{n!k!}} H_{nk} = \mathrm{e}^{\frac{|u|^2+|v|^2}{2}}\, \mathrm{e}^{\overline{u}\mathcal{A}_-^* - u\mathcal{A}_-}\, \mathrm{e}^{v\mathcal{A}_+^* - \overline{v}\mathcal{A}_+} H_{00}, \quad u, v \in \mathbb{C}, \tag{7.1}$$

where use has been made of the following equalities:

$$\mathrm{e}^{\frac{|\alpha|^2}{2}}\, \mathrm{e}^{\alpha \mathcal{A}_+^* - \overline{\alpha}\mathcal{A}_+} = \mathrm{e}^{\alpha\mathcal{A}_+^*}\, \mathrm{e}^{-\overline{\alpha}\mathcal{A}_+} \qquad \text{and} \qquad \mathrm{e}^{\frac{|\alpha|^2}{2}}\, \mathrm{e}^{\overline{\alpha}\mathcal{A}_-^* - \alpha\mathcal{A}_-} = \mathrm{e}^{\overline{\alpha}\mathcal{A}_-^*}\, \mathrm{e}^{\alpha\mathcal{A}_-},$$

$\alpha \in \mathbb{C}$. Clearly (see (7.1)),

$$\mathcal{J}\eta_{\overline{u},v}^{\mathrm{bcs}} = \eta_{\overline{v},u}^{\mathrm{bcs}}. \tag{7.2}$$

Since the H_{nk} are the eigenfunctions of $\mathcal{H}_{\varphi} = \mathcal{H}^{\uparrow} - \mathcal{H}^{\downarrow} = 2\mathcal{H}_{\mathrm{int}}^{\uparrow}$, these are coherent states related to the interaction Hamiltonian, or alternatively, to the modular automorphism group (6.7). They satisfy the resolution of the identity

$$\int_{\mathbb{C}}\int_{\mathbb{C}} \left| \eta_{\overline{u},v}^{\mathrm{bcs}} \right\rangle\!\left\langle \eta_{\overline{u},v}^{\mathrm{bcs}} \right| \mathrm{d}\nu(\overline{u}, u)\, \mathrm{d}\nu(\overline{v}, v) = I_{L^2(\mathbb{C}, \mathrm{d}\nu(\overline{z},z))}, \tag{7.3}$$

where $\mathrm{d}\nu(\overline{z}, z)$ is the measure introduced in (5.14).

We denote, as before, the subspace of $L^2(\mathbb{C}, \mathrm{d}\nu(\overline{z}, z))$, consisting of holomorhic functions in the variable z, by $\mathfrak{H}_{\mathrm{hol}}$ and the subspace of antiholomorphic functions (i.e. holomorphic in $\overline{z}$) by $\mathfrak{H}_{\mathrm{a\text{-}hol}}$. As we have seen before, the subspace $\mathfrak{H}_{\mathrm{hol}}$ is spanned by the vectors $\{H_{00}, H_{01}, H_{02}, \ldots, H_{0n}, \ldots\}$ and $\mathfrak{H}_{\mathrm{a\text{-}hol}}$ is spanned by $\{H_{00}, H_{10}, H_{20}, \ldots, H_{n0}, \ldots\}$. Let $\mathbb{P}_{\mathrm{hol}}$ and $\mathbb{P}_{\mathrm{a\text{-}hol}}$ be the corresponding projection operators, so that $\mathbb{P}_{\mathrm{hol}} \cap \mathbb{P}_{\mathrm{a\text{-}hol}} = |H_{00}\rangle\langle H_{00}|$. We now define the (non-normalized) coherent states:

$$\eta_z := \eta_{0,z}^{\mathrm{bcs}} = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} H_{n0} \in \mathfrak{H}_{\mathrm{a\text{-}hol}}, \qquad \forall z \in \mathbb{C} \quad \Longrightarrow \quad \eta_z(\overline{w}) = \mathrm{e}^{\overline{w}z},$$

$$\check{\eta}_{\overline{z}} := \eta_{\overline{z},0}^{\mathrm{bcs}} = \sum_{n=0}^{\infty} \frac{\overline{z}^n}{\sqrt{n!}} H_{0n} \in \mathfrak{H}_{\mathrm{hol}}, \qquad \forall \overline{z} \in \mathbb{C} \quad \Longrightarrow \quad \check{\eta}_{\overline{z}}(w) = \mathrm{e}^{w\overline{z}}. \tag{7.4}$$

Note that this definition is consistent with the fact that the function $K(\overline{w}, z) = \mathrm{e}^{\overline{w}z}$, which is a reproducing kernel, can be written as (see (5.25) and (5.26)):

$$\mathrm{e}^{\overline{w}z} = \sum_{n=0}^{\infty} \frac{(\overline{w}z)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} H_{n0}(\overline{w}, w)$$

$$= \sum_{n=0}^{\infty} \frac{\overline{w}^n}{\sqrt{n!}} H_{0n}(\overline{z}, z). \tag{7.5}$$

In view of (7.2) we also have,

$$\mathcal{J}\eta_z = \breve{\eta}_{\bar{z}}, \tag{7.6}$$

and furthermore,

$$\eta_z = \mathrm{e}^{\frac{|z|^2}{2}} \, \mathrm{e}^{z\mathcal{A}_+^* - \bar{z}\mathcal{A}_+} H_{0,0}, \qquad \breve{\eta}_{\bar{z}} = \mathrm{e}^{\frac{|z|^2}{2}} \, \mathrm{e}^{\bar{z}\mathcal{A}_-^* - z\mathcal{A}_-} H_{0,0}. \tag{7.7}$$

Because of (7.6), we shall call the pair $\{\eta_z, \ \breve{\eta}_{\bar{z}}\}$ *conjugate coherent states*.

The following *resolutions of the identity*, on $\mathfrak{H}_{\text{hol}}$ and $\mathfrak{H}_{\text{a-hol}}$ are then readily established:

$$\int_{\mathbb{C}} |\eta_z\rangle\langle\eta_z| \, \mathrm{d}\nu(\bar{z}, z) = \mathbb{P}_{\text{a-hol}},$$
$$\int_{\mathbb{C}} |\breve{\eta}_{\bar{z}}\rangle\langle\breve{\eta}_{\bar{z}}| \, \mathrm{d}\nu(\bar{z}, z) = \mathbb{P}_{\text{hol}}. \tag{7.8}$$

Using this and (7.6), we also obtain

$$\mathcal{J}\mathbb{P}_{\text{a-hol}} = \int_{\mathbb{C}} |\breve{\eta}_{\bar{z}}\rangle\langle\eta_z| \, \mathrm{d}\nu(\bar{z}, z),$$
$$\mathcal{J}\mathbb{P}_{\text{hol}} = \int_{\mathbb{C}} |\eta_z\rangle\langle\breve{\eta}_{\bar{z}}| \, \mathrm{d}\nu(\bar{z}, z). \tag{7.9}$$

The first operator is a partial isometry between $\mathfrak{H}_{\text{a-hol}}$ and $\mathfrak{H}_{\text{hol}}$, while the second is the reverse isometry. Note, also that

$$\mathcal{J}\mathbb{P}_{\text{hol}}\mathcal{J} = \mathbb{P}_{\text{a-hol}}. \tag{7.10}$$

We also note that, since (5.16) and (5.18) imply that $\mathcal{A}_+ H_{n0} = \sqrt{n} H_{n-10}$, we get

$$\mathcal{A}_+\eta_z = z\eta_z, \qquad \mathcal{A}_-\breve{\eta}_{\bar{z}} = \bar{z}\breve{\eta}_{\bar{z}},$$

so that, putting

$$\mathbf{A} = \begin{pmatrix} \mathcal{A}_+ & 0 \\ 0 & \mathcal{A}_- \end{pmatrix}, \qquad \mathcal{Z} = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\eta}_{\mathcal{Z}} = \begin{pmatrix} \eta_z \\ \breve{\eta}_{\bar{z}} \end{pmatrix},$$

we have $\mathbf{A}\,\boldsymbol{\eta}_{\mathcal{Z}} = \mathcal{Z}\,\boldsymbol{\eta}_{\mathcal{Z}}$. Hence these *vector coherent states* are eigenvectors of the matrix annihilation operator $\mathbf{A}$ (see also [4]).

8. Conclusions

In the problem of the electron studied here, the Hamiltonian governing the motion had a pure point spectrum, with each level being infinitely degenerate. Moreover, the energy levels were equally spaced. It seems possible to consider more general Hamiltonians, again with pure point spectra and infinite degeneracies, but not equally spaced, and go through a similar analysis. This would lead to other Hilbert spaces of holomorphic and antiholomorphic functions and to mutually commuting von Neumann algebras generated by more general shift operators. Hence, a modular structure similar to that considered in this paper can be recovered. There are also indications of an interesting connection between the problem studied here and the recently studied non-commutative quantum mechanics, as described, for example, in [13]. Both these aspects will be considered in a paper which is now in preparation. Furthermore, the relation between our procedure to lift the degeneracy and the right choice of the wavefunction for the fractional quantum Hall effect is a very appealing problem which we plan to consider in the near future, together with other aspects of the model intimately related to the Hall effect rather than with the Landau levels. Also, as mentioned earlier, introducing an electric field, in a direction orthogonal to the magnetic field, or introducing a confining (e.g. a two-dimensional harmonic oscillator) potential for the electron could also lift the degeneracy. The resulting effect on the mathematical theory would be to modify the modular structure and the KMS state. This is another interesting issue which we intend to study in greater depth.

Acknowledgments

Appendix. Basics of von Neumann algebras

In this appendix we collect together some basic definitions and facts about von Neumann algebras. Details may, for example, be found in [17]. Let $\mathfrak{H}$ be a Hilbert space over $\mathbb{C}$. For our purposes it will be enough to assume that $\mathfrak{H}$ is separable, of dimension N, which could be finite or infinite. We denote by $\mathcal{L}(\mathfrak{H})$ the set of all bounded operators on $\mathfrak{H}$. Let $\mathfrak{A} \subseteq \mathcal{L}(\mathfrak{H})$ and $\mathfrak{A}'$ its commutant (i.e. the set of all elements of $\mathcal{L}(\mathfrak{H})$ which commute with every element of $\mathfrak{A}$). Let $\mathfrak{A}$ be closed under linear combinations, (operator) multiplication and conjugation (i.e. if $A \in \mathfrak{A}$ then its adjoint $A^* \in \mathfrak{A}$). If in addition $\mathfrak{A} = \mathfrak{A}''$, then $\mathfrak{A}$ is called a *von Neumann algebra*. It can then be proved that $\mathfrak{A}$ is a weakly closed set. A von Neumann algebra always contains the identity operator $I_{\mathfrak{H}}$ on $\mathfrak{H}$. The von Neumann algebra $\mathfrak{A}$ is called a *factor* if $\mathfrak{A} \cap \mathfrak{A}' = \mathbb{C}I_{\mathfrak{H}}$.

Let $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ be a bounded linear functional on $\mathfrak{A}$, which we denote by $\langle \varphi; A \rangle, A \in \mathfrak{A}$. We call φ a *state* on this algebra if it also satisfies the two conditions: (a) $\langle \varphi; A^*A \rangle \geqslant 0, \forall A \in \mathfrak{A}$ and (b) $\langle \varphi; I_{\mathfrak{H}} \rangle = 1$. The state φ is said to be *faithful* if $\langle \varphi; A^*A \rangle > 0$ for all $A \neq 0$. A state is said to be *normal* if and only if there is a density matrix ρ such that $\langle \varphi \mid A \rangle = \text{Tr}[\rho A], \forall A \in \mathfrak{A}$. It is called a *vector state* if there exists a vector $\phi \in \mathfrak{H}$ such that $\langle \varphi; A \rangle = \langle \phi \mid A\phi \rangle, \forall A \in \mathfrak{A}$. Clearly, such a state is also normal. A vector $\psi \in \mathfrak{H}$ is called *cyclic* for the von Neumann algebra if the set $\{A\psi \mid A \in \mathfrak{A}\}$ is dense in $\mathfrak{H}$; it is said to be *separating* for $\mathfrak{A}$ if $A\psi = B\psi, A, B \in \mathfrak{A}$, if and only if $A = B$. It can then be shown that ψ is cyclic for $\mathfrak{A}$ if and only if it is separating for $\mathfrak{A}'$ and vice versa. An *automorphism* of the von Neumann algebra is a map $\alpha : \mathfrak{A} \longrightarrow \mathfrak{A}$ which preserves its algebraic structure. It can then be shown that α is norm preserving.

The centralizer of the von Neumann algebra $\mathfrak{A}$, with respect to the state φ, is the von Neumann subalgebra,

$$\mathfrak{M}_\varphi = \{B \in \mathfrak{A} \mid \langle \varphi; [B, A] \rangle = \langle \varphi; BA - AB \rangle = 0, \forall A \in \mathfrak{A}\}. \tag{A.1}$$

A von Neumann algebra is generated by all the unitary elements in it. As an example, if we take the unitary Weyl operators (see (4.1)) $U_\pm(x, y) = \exp[-\text{i}(xQ_\pm + yP_\pm)]$, $x, y \in \mathbb{R}$, with $Q_\pm, P_\pm$ as in (4.8) and (4.11), they generate the two von Neumann algebras $\mathfrak{A}_\pm$ introduced at the end of section 4. Similarly, a von Neumann algebra is generated by all the projection operators contained in it. Thus, as mentioned in section 3.4, the centralizer algebra $\mathfrak{M}_\varphi$ is generated by the projection operators $\mathbb{P}_i^\ell$, $i = 1, 2, \ldots, N$.

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